Interval observer design based on nonlinear hybridization and practical stability analysis

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SUMMARY

In this paper we design an interval observer for nonlinear uncertain continuous-time dynamical systems, in the unknown-but-bounded error framework. We show how to generate two coupled hybrid observers, which yield the best conservative upper and lower component-wise bounds for the state, provided bounds are available for initial conditions and uncertainties. The interval observer depends on a tuning gain, which is chosen in order to satisfy the practical stability of a given hybrid dynamical system. An example is studied to emphasize the performance of our method.

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1. INTRODUCTION

State estimation with uncertain nonlinear continuous-time dynamical systems is an issue of importance in many fields. Classical point state estimation algorithms such as the extended Luenberger observer, the extended Kalman filter [1, 2], the high gain observer [3] and the sliding mode observer [4, 5] are more or less robust with respect to disturbances and measurement noises, however they often fail to give satisfactory estimations in the presence of uncertainties in model parameters [6].

When all uncertain quantities, measurement noise, modeling error, model and input uncertainties, remain within a bounded set with known bounds, the estimation problem no longer has an unique solution and must be tackled with set-membership estimation techniques. Clearly, if the solution set can be accurately estimated on-line, it would undoubtedly help developing new robust control and diagnosis methods.

In the literature there are two approaches for set-membership state estimation (SME). They aim at providing sets containing all the state vectors consistent with the model structure, uncertainties in initial

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state and parameters and also with the experimental data and their feasible domains. These approaches are founded on reachability computation for uncertain nonlinear systems using novel arithmetics on geometrical representations such as ellipsoids [7, 8], paralleloptopes [9], zonotopes [10], boxes or interval vectors [11]. In this paper, we will consider the latter representation.

The first type of SME approach addresses the case of state estimation from discrete-time measurement and was introduced in [12] and improved by [13, 14, 15, 16, 17, 18]. It relies on two steps: (i) a prediction step, which consists in computing an over-approximation for the reachable set of the uncertain nonlinear system by using set integration methods for uncertain differential equations [19, 20, 21, 22]; (ii) a correction step, which uses consistency techniques to prune at each measurement time instant inconsistent state vectors from the conservative estimation of the reached state set [11]. In the sequel of the paper, this approach will not be considered.

The second type of SME approach deals with the case of state estimation from continuous-time measurement. It was proposed for the first time in [23, 24] to address a particular class of uncertain systems, namely bioreactors. The main concept of this approach consists in designing an interval observer which yields the best conservative upper and lower component-wise bounds for the state provided known bounds for initial conditions and uncertainties are available. Thereafter, several techniques were developed to improve the performance of this interval observer such as using linear transformation, running in parallel bundle of interval observers and realizing regular re-initialization [25, 26, 27, 28].

The purpose of this paper is to introduce a generic method for designing interval observers for a broad class of uncertain nonlinear systems. Our method is based on (i) an enhanced version of the hybrid bounding method for computing over-approximation of the reachable set of uncertain nonlinear systems [22]; and on (ii) a suitable choice of the observation gain matrix to guarantee the convergence of the observation error, namely keeping the estimated state trajectories within a given tight bounded box. To prove the latter point, we will use the notion of practical stability for hybrid and switching systems developed in [29]. Note that, our design method does not need the cooperative property of the linear part of the proposed interval observer nor the boundedness of its nonlinear part, contrariwise to the methods available in the literature. However, we assume that the nonlinear part is Lipschitz continuous with respect to the state vector and also with respect to the uncertain parameters vector.

The paper is divided into four sections. In Section 2, we briefly recall the bounding method for uncertain nonlinear systems, then we state in Section 3 our main result regarding the design of interval observers. An illustrative example is given in Section 4 to show the potential of the proposed hybrid interval observer. Appendix II recalls Müller’s theorem. Finally, in Appendix III, we prove the main result of the paper using the practical stability concept, and we show how to compute a suitable observation gain matrix.

2. BOUNDING METHOD FOR UNCERTAIN NONLINEAR SYSTEMS
The main techniques used by our interval observer design approach is the hybrid bounding method for computing an over-approximation of the state flow generated by uncertain nonlinear dynamical systems [22, 30]. It is recalled in this section.
2.1. Bounding using Müller’s theorem

Consider the nonlinear uncertain continuous-time dynamical system

$$\dot{x} = f(x, p, t), \quad x_0 \in [\bar{x}_0, \underline{x}_0] \subset \mathbb{D} \subseteq \mathbb{R}^n, \quad p \in [\bar{p}, \underline{p}] = \mathbb{P} \subset \mathbb{R}^{n_p} \quad \text{and} \quad t \geq 0. \quad (1)$$

The vector field $f(\ldots)$ is continuous over the domain $\mathbb{D} \times \mathbb{P} \times \mathbb{R}^+$. The initial state vector and the uncertain parameters vector are defined as interval vectors [11] with dimensions $n$ and $n_p$ respectively.

The core idea of Müller’s theorem [31, 32], which is given in Appendix II for completeness, consists in building a lower and an upper coupled time-continuous dynamical systems which involve no uncertainty and enclose in a guaranteed way all the state trajectories generated by the original uncertain system (1). In the sequel, we will write the coupled deterministic bounding systems as follows

$$\begin{cases}
\dot{x} = f(x, x, \underline{p}, \underline{p}, t) \\
\dot{\bar{x}} = f(\bar{x}, \bar{x}, \bar{p}, \bar{p}, t) \\
\bar{x}(t_0) = \underline{x}_0, \\
\bar{x}(t_0) = \underline{x}_0
\end{cases} \quad (2)$$

where $\underline{x}(t)$ denotes the minimal solution and $\bar{x}(t)$ the maximal solution of (1).

The question which arises now is how to obtain the bounding systems (2) from the uncertain system (1). In fact, when all the components of $f(\ldots)$ are monotonic with respect to each uncertain parameter $p_k$ and each state variable $x_j$, it is quite easy to obtain these bounding systems as proposed in [32].

**Rule 1 (Monotonicity test) [32]:** For instance, in order to build the upper system, that is the one which yields the maximal solution $\bar{x}(t)$, one must replace in the formal expression of $f_i(\ldots), \ i \in \{1, \ldots, n\}$.

1. $x_i$ by $\underline{x}_i$,
2. $x_j$ by $\bar{x}_j$ if $\frac{\partial f_i(\ldots)}{\partial x_j} \geq 0$ or by $\underline{x}_j$ if $\frac{\partial f_i(\ldots)}{\partial x_j} < 0, \ j \in \{1, \ldots, n\}$ and $j \neq i$
3. $p_k$ by $\bar{p}_k$ if $\frac{\partial f_i(\ldots)}{\partial p_k} \geq 0$ or by $\underline{p}_k$ if $\frac{\partial f_i(\ldots)}{\partial p_k} < 0, \ k \in \{1, \ldots, n_p\}$.

The components of the lower system, that is the one which yields the lower solution $\underline{x}(t)$ are derived by reversing monotonicity conditions.

A detailed presentation of the use of this rule can be found in [33, 22]. Nevertheless, to illustrate its use, let us consider the uncertain linear system

$$\dot{x} = \begin{bmatrix} -1 & 0.8 \\ -4 & 2 \end{bmatrix} x + \begin{bmatrix} p \\ 0 \end{bmatrix} u(t) \quad (3)$$

where $u(t)$ is an unit step ($u(t) = 1, \ t \geq t_0$). The initial state vector $\bar{x}(t_0) \in [\underline{x}(t_0), \bar{x}(t_0)] \subset \mathbb{D} = \mathbb{R}^2$ and the uncertain parameter $p \in [\underline{p}, \bar{p}] \subset \mathbb{R}$. For this system, it is clear that all the partial derivatives with respect to the state variables and the uncertain parameter have constant sign. Then, we can use Rule 1 to obtain the following coupled systems that yield bounding systems for (3)

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\bar{x}} \end{bmatrix} = \begin{bmatrix} -1 & 0.8 & 0 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & -1 & 0.8 \\ -4 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \bar{x} \end{bmatrix} + \begin{bmatrix} \underline{p} \\ 0 \\ p \\ 0 \end{bmatrix} u(t), \quad (4)$$
Remark 1. When the systems under study are order-preserving monotone dynamical systems [34], we can build the lower and the upper systems separately [30]. Thus, the bounding systems are decoupled and can be written as follows

\[
\begin{align*}
\dot{x} &= \bar{f}(\bar{x}, \bar{p}, p, t), \quad \bar{x}(t_0) = \bar{x}_0, \\
\dot{x} &= \hat{f}(\hat{x}, \hat{p}, p, t), \quad \hat{x}(t_0) = \hat{x}_0.
\end{align*}
\]

(5)

In next section, we show how to use Rule 1 for bounding systems for which the sign of some partial derivatives change within the reachable set.

2.2. Bounding using a nonlinear hybridization

In practice, one does not expect the signs of the partial derivatives of nonlinear systems to remain constant over the reachable set. Therefore, it is of interest to partition the state space \(\mathbb{D}\) of the uncertain system – in fact it is not necessary to partition whole state space, but an over-approximation of the reachable set only – into cells that only intersect on their boundaries, and within which the partial derivatives have constant signs. Then, one can use Rule 1 over each cell to build suitable bounding systems. Finally, as time spans \([t_0, t_f]\), the collection of these bounding systems can be used to generate an over-approximation of the set reachable by the original uncertain system. In other words, this nonlinear hybridization method for reachability computation introduced in [22] uses a switching system \(\mathcal{H}\) composed of a collection of time-continuous bounding dynamical systems

\[
\mathcal{H} = \{(M_1, \mathbb{D}_1), (M_2, \mathbb{D}_2), \ldots, (M_q, \mathbb{D}_q), \ldots\}, \quad q \in \{1, \ldots, l\}
\]

(6)

where each subsystem \(M_q\) is obtained by using Rule 1 on each domain \(\mathbb{D}_q\). They are described by the following state representation

\[
M_q \equiv \left\{ \begin{array}{l}
\bar{x}(t) = \bar{L}_q(\bar{x}, x, p, \bar{p}, t) \\
\hat{x}(t) = \hat{L}_q(\hat{x}, x, p, \hat{p}, t)
\end{array} \right\} \text{ for } q \in \{1, \ldots, l\}.
\]

(7)

The switching sequence \(\sigma = \{(t_0, q_0), (t_1, q_1), \ldots, (t_s, q_s), \ldots\}\) (where \(q_t \in L\) and \(t_0 \leq t_1 \leq \cdots \leq t_s \leq \ldots\) ) of the switching system \(\mathcal{H}\) that specifies that the local bounding system \(M_{q_t}\) is active during \([t_s, t_{s+1})\) is generated by the sign change of the partial derivatives

\[
\begin{align*}
C_{p_{(i,k)}}(x, p, t) &= \frac{\partial f_i(x, p, t)}{\partial p_k}, \quad i \in \{1, \ldots, n\}, \quad k \in \{1, \ldots, n_p\} \\
C_{x_{(i,j)}}(x, p, t) &= \frac{\partial f_i(x, p, t)}{\partial x_j}, \quad i, j \in \{1, \ldots, n\}, \quad i \neq j.
\end{align*}
\]

(8)

These partial derivatives act as guard conditions that drive the discrete transitions from one bounding system to another one. For each discrete transition corresponds a reset function given by

\[
(\bar{x}(t_i), \hat{x}(t_i)) = J_{q_{i-1}, q_i}(\bar{x}(t^-_i), \hat{x}(t^-_i)) = (\bar{x}(t^-_i), \hat{x}(t^-_i)).
\]

(9)

We can now define the state-dependent switching law \(\gamma\) over time interval \(\tau = [t_0, t_f]\) as follows

Definition 1 (State-dependent switching law \(\gamma\)) Given a time interval \(\tau\), a switching law \(\gamma\) over \(\tau\) is a mapping: \(\mathbb{R}^n \times \mathbb{R}^{n_p} \rightarrow \Sigma\) which specifies a nonZeno switching sequence \(\sigma \in \Sigma\) for any boxes of initial state \([x_0]\) and uncertain parameters \([p]\). Here \(\Sigma\) is the set of switching sequences \(\sigma\) over \(\tau\).

To summarize, for a given initial state box \([x_0]\) and interval vector of uncertain parameters \([p]\), the execution of the hybrid deterministic automaton (6) characterizes a conservative over-approximation of the reachable set of the nonlinear continuous-time dynamical system (1).
2.3. Practical bounding methods

When one uses our hybrid bounding approach with uncertain dynamical systems, one often meets situations where for some time instants, the sign of some partial derivatives (8) are not constant over the over-approximation \([\mathbf{x}(t), \dot{\mathbf{x}}(t)]\) of the reachable set, hence Rule 1 cannot be used. In the sequel, we introduce propositions that can be used to overcome this issue.

Let us consider the components \(f_i(., z_{j,..})\) for which the range of \((\partial f_i/\partial z_j)(., z_{j,..}) = C z_{(i,j)}(., z_{j,..})\) over \([z_j, \bar{z}_j]\) contains zero, where \(i \in \{1, \ldots, n\}\) and \(z_j\) denotes either a state variable \(x_j\) \(i \neq i\) or an uncertain parameter \(p_k\).

**Property 1 (P1)** ([18] and the references therein) Any Lipschitz function \(f_i\) can be written as the sum of one decreasing function \(d_i(., z_{j,..})\) and one increasing function \(c_i(., z_{j,..})\), i.e.:

\[
\forall z_j \in [\underline{z}_j, \bar{z}_j] = [z_j], \quad f_i(., z_{j,..}) = d_i(., z_{j,..}) + c_i(., z_{j,..})
\]  

(10)

Hence,

\[
\forall z_j \in [\underline{z}_j, \bar{z}_j], \quad d_i(., z_{j,..}) + c_i(., z_{j,..}) \leq f_i(., z_{j,..}) \leq d_i(., \bar{z}_j, ..) + c_i(., \bar{z}_j, ..).
\]  

(11)

In fact, there are several ways of decomposing function \(f_i\). Some decompositions may introduce large overapproximation when writing inequalities (11), hence very loose bounds for the computed reachable set. Therefore, we must look for decompositions that do not introduce the latter over-approximations, decompositions which are obtained using what we define as a pragmatic way. In other words, to obtain the inequalities (11), one would, for instance, analyze the monotonicity of some subexpressions of \(f_i\) instead of using Lipschitz constants. Furthermore, it is also interesting to investigate the possibility to decompose, in a pragmatic way, function \(f_i\) as the product of one increasing function and one decreasing one, which we summarize in the following hypothesis (H1).

**Hypothesis 1 (H1)** Function \(f_i(., z_{j,..})\) can be written in a pragmatic way as the product of one decreasing function \(d_i(., z_{j,..})\) and one increasing function \(c_i(., z_{j,..})\), viz.

\[
\forall z_j \in [\underline{z}_j, \bar{z}_j], \quad f_i(., z_{j,..}) = d_i(., z_{j,..}) \times c_i(., z_{j,..}).
\]  

(12)

**Proposition 2 (P2)** If (H1) is true, then according to the sign of \(d_i(., z_{j,..})\) and \(c_i(., z_{j,..})\) on the bounded interval \([\underline{z}_j, \bar{z}_j]\), it is always possible to frame \(f_i(., z_{j,..})\) by two elements of the following set,

\[
\mathcal{B} = \{d_i(., z_{j,..}) \times c_i(., z_{j,..}); \quad d_i(., z_{j,..}) \times c_i(., z_{j,..}); \quad d_i(., z_{j,..}) \times c_i(., z_{j,..}); \quad d_i(., z_{j,..}) \times c_i(., z_{j,..}); \quad \min(f_i(., z_{j,..}), f_i(., \underline{z}_j, ..))\}
\]  

(13)

where \(z_j', z_j''\) belong to \([\underline{z}_j, \bar{z}_j]\) if \(c_i(., z_{j,..})\) and \(d_i(., z_{j,..})\) do not have constant sign on \([\underline{z}_j, \bar{z}_j]\), and satisfy respectively \(c_i(., z_j') = 0\), \(d_i(., z_j'') = 0\).

**Proof.** Consider the case where the sign of \(d_i(., z_{j,..})\) and \(c_i(., z_{j,..})\) is not constant on the bounded interval \([\underline{z}_j, \bar{z}_j]\). Then, since \(c_i(., z_{j,..})\) is increasing and \(d_i(., z_{j,..})\) is decreasing it is clear that the minimal value of \(f_i(., .\) is negative and given by

\[
f_i = \min\{c_i(., z_{j,..}) \times d_i(., z_{j,..}), c_i(., z_{j,..}) \times d_i(., z_{j,..})\} = \min\{f_i(., z_{j,..}), f_i(., \underline{z}_j, ..)\}
\]  

(14)

Now, to obtain the maximal value of \(f_i(., .\) which is obviously positive, let us characterize the domain \([z_j'] \in [\underline{z}_j]\) over which \(f_i(., .\) is positive. To do so, we must compute \(z_j'\) and \(z_j''\) for which \(c_i(., z_j') = 0\) and
and \( d_i(., z_{j'}', .) = 0 \). Then we obtain \([z_j]' = [z_{j'}', z_{j''}]'\) or \([z_j]' = [z_{j''}, z_j']\). Thus, since \( c_i(., z_{j', .}) \) and \( d_i(., z_{j', .}) \) have the same sign on \([z_j]'\), by direct computation we can upper bound \( f_i(., z_{j', .}) \) by

\[
f_i(., z_{j', .}) \leq d_i(., z_{j', .}) \times c_i(., z_{j''}, .) = \overline{f}_i.
\]

(15)

Hence (14) and (15) belong to \( \mathcal{B} \). To complete the proof of the proposition we can use a similar reasoning to show that the set \( \mathcal{B} \) contains all the possible lower and upper bounds of \( f_i(., z_{j', .}) \) whatever the signs of \( d_i(., z_{j', .}) \) and \( c_i(., z_{j', .}) \).

Next example will illustrate the use of property P1 and proposition P2 within our hybrid bounding approach.

Example 1. Consider the nonlinear continuous-time dynamical system

\[
\begin{align*}
x_1 &= x_2, \\
x_2 &= -2x_1^2 \exp(x_2 - 1)
\end{align*}
\]

(16)

where initial state vector is uncertain \((x_1(t_0), x_2(t_0)) \in [x_1(t_0)] \times [x_2(t_0)] \subset \mathbb{R}^2\). To characterize in a guaranteed way the reachable set, we will build the hybrid automaton according to the sign of the partial derivatives. For all \( x \in \mathbb{R}^2 \), we have \( Cx_{1,2,x}(x) = \partial x_1/\partial x_2 = 1, Cx_{2,1,x}(x) = \partial x_2/\partial x_1 = -4x_1 \exp(x_2 - 1), \) and \( \text{sign}(Cx_{2,1,x}(x)) = -\text{sign}(x_1) \). Then, given a time instant \( t \), we can build the following locally bounding systems, according to the sign of \( Cx_{2,1,x}(x) \) on \([x_1](t)\):

- If for all \( x_1 \in [x_1](t) \), \( x_1 < 0 \) then \( \text{sign}(Cx_{2,1,x}(x)) \) is positive. By using the monotonicity test given in Rule 1 we obtain the first bounding system \( M_1 \),

\[
M_1 = \begin{cases} 
\dot{x}_1 = x_2 \\
\dot{x}_2 = -2x_1^2 \exp(x_2 - 1)
\end{cases}
\]

(17)

- If for all \( x_1 \in [x_1](t) \), \( x_1 > 0 \) then \( \text{sign}(Cx_{2,1,x}(x)) \) is negative and thanks to Rule 1 we obtain the second bounding system \( M_2 \),

\[
M_2 = \begin{cases} 
\dot{x}_1 = x_2 \\
\dot{x}_2 = -2x_1^2 \exp(x_2 - 1)
\end{cases}
\]

(18)

- Now, if \( 0 \in [x_1](t) \) the sign of \( Cx_{2,1,x}(x) \) changes on \(([x_1](t), [x_2](t))\) and Rule 1 cannot be used. We can use Proposition 2 to frame \( \dot{x}_2 \). First, we write \( \dot{x}_2 = d_2(x_1) \times c_2(x_1, x_2) \), where \( c_2(x_1, x_2) = x_1 \exp(x_2 - 1) \) is an increasing function of \( x_1 \) and \( d_2(x_1) = -2x_1 \) is a decreasing function of \( x_1 \). Second, since \( \dot{x}_2 \) is negative, the signs of \( c_2(., .) \) and \( d_2(.) \) are not constant over \(([x_1](t), [x_2](t))\), and \( c_2(0, x_2) = d_2(0) = 0 \), we can use the following double inequality to frame \( \dot{x}_2 \) on \([x_1](t)\) as suggested in Proposition 2: \( \forall (x_1, x_2) \in [x_1](t) \times [x_2](t) \), \( -2 \min\{x_1^2, x_2^2\} \exp(x_2 - 1) \leq \dot{x}_2 \leq 0 \). Finally, thanks to Rule 1 and Proposition 2, we derive the third bounding system \( M_3 \) as follows:

\[
M_3 = \begin{cases} 
\dot{x}_1 = x_2 \\
\dot{x}_2 = 0 \\
\dot{x}_1 = x_2 \\
\dot{x}_2 = -2\min\{x_1^2, x_2^2\} \exp(x_2 - 1)
\end{cases}
\]

(19)
Hence, for any initial state vector box \([\bar{x}](t_0)\), an over-approximation of the reachable set of (16) is obtained by running the hybrid system \(\mathcal{H} = \{M_1, M_2, M_3\}\) governed by the automaton depicted on Figure 1. Each bounding system \(M_\ell, \ell \in \{1, 2, 3\}\), is activated on specific state region delimited by the guard condition \(C_{(\ell,3)}(x)\) as illustrated on Figure 2. On Figure 3, a zoom around the guard condition is depicted to illustrate the fact that bounding system \(M_3\) is necessary to cross the guard set.

Finally, note that when one uses our hybrid bounding method, the cardinality of the collection of all possible bounding dynamical systems (as defined by (6)) depends on the number of partial derivatives that can change sign onto the reachable set. When the sign of a partial derivative over the over-approximation \([\bar{x}(t), \bar{x}(t)]\) of the reachable set is either positive or negative, Rule 1 can be used, but when the partial derivative changes sign over \([x(t), \bar{x}(t)]\), property P1 or proposition P2 must be used. There are three cases, hence the cardinality of \(\mathcal{H}\)

\[ l = \text{card}(\mathcal{H}) = 3^d, \]

where \(d\) is the number of these partial derivatives.

3. HYBRID INTERVAL OBSERVER

Consider the uncertain continuous-time dynamical system described by the following nonlinear state equations

\[
\begin{aligned}
\dot{x} &= A(t)x + \Phi(x, p, t) \\
y &= C(t)x \\
p &\in [p] \\
x(t_0) &\in [x_0]
\end{aligned}
\]  

(20)

where \(A(t)\) and \(C(t)\) are matrices with dimension \(n \times n\) and \(m \times n\) respectively, and where \(n, m\) and \(n_p\) are respectively, the dimensions of the state vector \(x \in \mathbb{R}^n\), the output vector \(y \in \mathbb{R}^m\) and the uncertain parameter vector \(p \in [p] \subset \mathbb{R}^{n_p}\). \(\Phi(\ldots, \ldots)\) is a \(n\)-dimensional vector of Lipschitz continuous functions (w.r.t both parameter and state variables) defined on \(\mathbb{R}^n \times \mathbb{R}^{n_p} \times \mathbb{R}^+\).

Remark 2. In practice, formulation (20) is not restrictive, since we can rewrite without loss of generality a given nonlinear equation \(\dot{x} = f(x, p, t)\) as

\[ f(x, p, t) = A(t)x + (f(x, p, t) - A(t)x) = A(t)x + \Phi(x, p, t). \]  

(21)

For instance, matrix \(A(t)\) can be taken as the gradient \(\nabla f(x_1, p_1, t)\) where \(x_1(t)\) is the trajectory solution of \(\dot{x} = f(x, p, t)\) when \(x(t_0) = \text{Mid}([x_0])\) and \(p_1 = \text{Mid}([p])\). We denote by \(\text{Mid}(\cdot)\) the midpoint of an interval vector \([11]\). Moreover, this formulation encloses situations where the state equations depend of some uncertain inputs which can be viewed as uncertain parameters.

We assume that measurements \(y_M(t)\) are subject to additive errors, that are unknown but bounded with known bounds. Thus the feasible domains for measurements are given by the following boxes

\[ Y(t) = [y_M(t) - b(t), y_M(t) + b(t)] \]  

(22)

where \(b(\cdot)\) are vectors of nonnegative entries \(b_j(\cdot), j \in \{1, \ldots, m\}\), that correspond to a priori known measurement errors bounds.

Here we give some matrix definitions useful to state our main result.
**Definition 2.** For an arbitrary real $n \times n$ matrix $A$ we let $\tilde{A}$ be the $n \times n$ Metzler matrix (i.e. with all off-diagonal elements non-negative), defined by

\[
\tilde{a}_{ij} = \begin{cases} 
  a_{ij}, & \text{if } i = j \\
  |a_{ij}|, & \text{if } i \neq j.
\end{cases}
\]

**Definition 3.** Let $A$ be an arbitrary square real matrix of order $n$. We denote by $\mathcal{A}$ its associated square matrix of order $2n$, obtained when applying the two first points of Rule 1 to the dynamical system $\dot{x} = Ax, \ x_0 \in [x_0]$, viz.

\[
(\tilde{x}, \tilde{x})^T = \mathcal{A}(\tilde{x}, \tilde{x})^T.
\]

To be clear, reconsider system (3) of subsection 2.1. Then, when we apply the two first items of Rule 1 with the square matrix $A = \begin{bmatrix} -1 & 0.8 \\ -4 & 2 \end{bmatrix}$, we obtain

\[
\mathcal{A} = \begin{bmatrix} -1 & 0.8 & 0 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & -1 & 0.8 \\ -4 & 0 & 0 & 2 \end{bmatrix}.
\]

**Definition 4.** Let $K$ be an $n \times m$ arbitrary real matrix. Denote by $\mathcal{K}$ its associated real matrix with dimension $2n \times 2m$ obtained when we use the third item of Rule 1 to bound the algebraic system $z = -Ky$ when $y \in [y]$, viz.

\[
(z, \tilde{z})^T = \mathcal{K}(\tilde{y}, \tilde{y})^T.
\]

Now, let (26) be a Luenberger-like observer for system (20) when we assume that its parameters are known perfectly and its measurements are exact,

\[
\begin{align*}
\dot{\hat{x}} &= A(t)\hat{x} + \Phi(\hat{x}, p, t) + K(t)(\tilde{y}(t) - y(t)) \\
\tilde{y} &= C(t)\hat{x}
\end{align*}
\]

where $K(.)$ is the observation gain matrix with dimension $n \times m$. Hence, it is clear that for any matrix $K(.)$, any $x_0 \in [x_0]$ and any $p \in [p]$, the solution of system (20) is also a solution of the Luenberger-like observer (26). By using our nonlinear hybridization bounding method we can derive the hybrid observer (27) that generates a guaranteed enclosure of all the possible state trajectories of the uncertain nonlinear system (20).

\[
\begin{bmatrix} 
\dot{\hat{x}} \\
\tilde{\hat{x}} 
\end{bmatrix} = \mathcal{A}(t)\begin{bmatrix} \hat{x} \\
\tilde{x} \end{bmatrix} + \mathcal{G}(t)\begin{bmatrix} \hat{x} \\
\tilde{x} \end{bmatrix} + \mathcal{K}(t)\begin{bmatrix} \tilde{y}_M(t) \\
\tilde{y}_M(t) \end{bmatrix},
\]

where $q$ is taken in the set $L = \{1, \ldots, l\}$, which contains the indexes of the continuous modes constituting the hybrid system (27). Matrices $\mathcal{A}(.)$ and $\mathcal{G}(.)$ are of dimension $2n \times 2n$, built respectively from matrices $A(.)$ and $K(.)C(.)$ according to definition 3. Also, matrices $\mathcal{K}(.)$ are of dimension $2n \times 2m$, built from matrix $-K(.)$ according to definition 4.

We can now state our main result.
Theorem 1. For all \( q \in L \), define \( E_q(t) = A(t) + K_q(t)C(t) \), and \( \lambda_0(t) = \max\{\lambda_q(t), 0\} \) a piecewise \( 0 \) \( \int_{t_0}^{t_f} \lambda_0(z)dz < \frac{1}{2} \), \( (28) \) then the hybrid system (27) is an interval observer for the uncertain system (20) with the following properties satisfied \( \forall x(t_0) \in [\xi(t_0), \bar{x}(t_0)] \), \( \forall p \in [\underline{p}, \bar{p}] \), \( \forall t \geq t_0 \).

\begin{align*}
\text{Positivity: } & (x(t) - \bar{x}(t) \geq 0) \land (\bar{x}(t) - x(t) \geq 0) \\
\text{Convergence: } & \lim_{t \rightarrow +\infty} \|x(t) - \bar{x}(t)\| = w(\|\underline{p} - \bar{p}\|, \|b\|) \\
\end{align*}

(29) (30)

where \( w(\ldots) \) is a scalar increasing function w.r.t. to both the width of the parameters uncertainty box, and measurements errors bounds. Here \( \|\cdot\| \) denotes the maximal norm of vectors.

Proof. see Appendix III.

In summary, we have given a condition that ensures that the hybrid interval observer (27) yields on-line, an upper \( \bar{x}(t) \) and a lower \( \underline{x}(t) \) state trajectories that enclose in guaranteed way all the possible state vectors generated by the uncertain system (20) and consistent with the feasible domains for experimental data. Note that our interval observer design method needs not the matrices \( A(\cdot) + K_q(\cdot)C(\cdot) \) to be cooperative nor the nonlinear function \( \Phi(\ldots) \) to be bounded. These relaxations are of importance and should allow the design of interval observers for a broader class of uncertain nonlinear systems, including those considered in [23, 24, 26, 27, 35].

4. EXAMPLE

As an illustrative example, we consider a three-states biochemical reactor given in [36], where the state variables are the biomass \( x_1 \), the substrate \( x_2 \) and the product \( x_3 \). The process can be modeled by the following ordinary differential equations (31) and the output equations (32):

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
-D & 0 & 0 \\
0 & -D & 0 \\
\beta & 0 & -D
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
\mu(x_2,x_3)x_1 \\
Dx_2 - \frac{\mu(x_2,x_3)x_1}{\theta} \\
a\mu(x_2,x_3)x_1
\end{bmatrix}
\]

(31)

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

(32)

where the specific growth rate \( \mu(\ldots) \) is a function of both the substrate and the product concentrations

\[ \mu(x_2,x_3) = \frac{\mu_m(1 - (x_3/\bar{x}_3))x_2}{k_x + x_2}. \]

(33)

We assume that the maximum growth rate \( \mu_m \), the saturation parameter \( k_x \) and the input substrate \( x_{2f} \) are unknown but belong to the following intervals \([0.35, 0.45] \) h\(^{-1}\), \([1.4, 1.6] \) g/L and \([17, 23] \) g/L respectively. We denote \((p_1,p_2,p_3)^T = (\mu_m,k_x,x_{2f})^T\). The initial state variables are also unknown.
Remark 3. Note that we need not to assume matrix $\mathbf{A}$ such that $\text{det} \mathbf{A} = 0$, as functions of matrix $\mathbf{K}$ entries. We have

$$
\text{det} | \mathbf{A} - \hat{\mathbf{A}} | = \text{det} \begin{vmatrix}
\lambda + D & -k_1 & -k_4 \\
0 & \lambda + D - k_2 & -k_3 \\
-\beta & -k_3 & \lambda + D - k_6
\end{vmatrix}
= (\lambda + D)((\lambda + D - k_2)(\lambda + D - k_6) - |k_3||k_5|) - \beta |k_1||k_3| + (\lambda + D - k_2)|k_4|
$$

Then, if we set $k_4 = k_5 = 0$ we obtain the following eigenvalues for $\hat{\mathbf{A}}$

$$
\lambda_1 = -D, \quad \lambda_2 = k_2 - D, \quad \lambda_3 = k_6 - D.
$$

It is obviously sufficient to take $k_2 < 0$ and $k_6 < 0$ so that (28) is satisfied. Indeed, for $k_2 < 0$ and $k_6 < 0$, we have $\lambda_0 = \max \{ \lambda_1, \lambda_2, \lambda_3, 0 \} = 0$, and hence (28) is always true, i.e. $\int_0^T \lambda_0 \, dz = 0 < \frac{1}{2}$.

**Remark 3.** Note that we need not to assume matrix $\mathbf{A} + \mathbf{KC}$ to be cooperative to build the interval observer (42). Hence, $k_1$ and $k_3$ can take any value in $\mathbb{R}$.

**4.1. Building the bounding systems.** Obviously, we can deal with the linear part and the nonlinear part of the bioreactor separately to build the bounding systems. We start by bounding the linear part of bioreactor (31). To do so, we use Rule 1 and we obtain

$$
[\alpha + \mathcal{C}] = \begin{bmatrix}
-D & k_1 & 0 & 0 & 0 & 0 \\
0 & -D + k_2 & 0 & 0 & 0 & 0 \\
-\beta & 0 & -D + k_6 & 0 & k_3 & 0 \\
0 & 0 & 0 & -D & k_1 & 0 \\
0 & 0 & 0 & -D + k_2 & 0 & k_3 \\
0 & k_3 & 0 & -\beta & 0 & -D + k_6
\end{bmatrix}
$$

Now, to build bounding functions for the nonlinear part of the bioreactor (31)

$$
\Phi(x, p) = \begin{bmatrix}
\mu(x_2, x_3) x_1 \\
D x_2 f - \frac{\mu(x_2, x_3) x_1}{Y} \\
\alpha \mu(x_2, x_3) x_1
\end{bmatrix}
$$
we must analyze the signs of its partial derivatives with respect to the state variables and uncertain parameters on the state space. Note that, for this example, the state variables $x_i$, $i \in \{1, 2, 3\}$, are always positive because for $x_i = 0$, we have $\dot{x}_i \geq 0$, namely the faces $x_i = 0$ are repulsive. For example, we have for $x_3 = 0$, $\dot{x}_3 = DX_3 > 0$. By direct computation we can show that the sign of the partial derivatives $(\partial \phi_1 / \partial x_2)(x, p)$, $(\partial \phi_1 / \partial p_1)(x, p)$, $(\partial \phi_2 / \partial p_2)(x, p)$, $(\partial \phi_3 / \partial x_1)(x, p)$, $(\partial \phi_3 / \partial x_2)(x, p)$, and $(\partial \phi_3 / \partial p_1)(x, p)$ is equal to the sign of $(1 - \frac{x_3}{x_{3m}})$. Similarly, the sign of the partial derivatives $(\partial \phi_1 / \partial p_1)(x, p)$, $(\partial \phi_2 / \partial x_1)(x, p)$, $(\partial \phi_3 / \partial p_2)(x, p)$, and $(\partial \phi_2 / \partial p_1)(x, p)$ is equal to the sign of $(\frac{x_3}{x_{3m}} - 1)$. Finally, the above partial derivatives signs depend on the sign of the unique expression $1 - \frac{x_3}{x_{3m}}$, for $x_3 \in [x_3](t)$. All the other partial derivatives have constant sign. Finally, because we have dealt with the linear and nonlinear parts separately, and because we have detected that partial derivatives signs are driven by a single function, we can reduce the cardinal number of the collection of continuous bounding dynamical systems of hybrid system (6) to $3^1 = 3$ instead of $3^9 = 19683$. Hence the nonlinear part of the bioreactor (31) has three bounding functions according to the sign of $1 - \frac{x_3(t)}{x_{3m}}$, for a given time instant $t$ in $[t_0, t_f]$: 

- If $(\forall x \in [x_3](t), \ x_3 \leq x_{3m})$ by using only Rule 1, we obtain as first bounding functions

$$
\Phi_1(x, x, p, p) = \begin{pmatrix}
\frac{\mu_m(1 - \frac{x_3}{x_{3m}})\sigma_2}{\xi_1 + \xi_2}

\frac{\mu_m(1 - \frac{x_3}{x_{3m}})\sigma_2}{\xi_1 + \xi_2} \Delta_1

\end{pmatrix}
$$

- If $(\forall x \in [x_3](t), \ x_3 > x_{3m})$ by using only Rule 1, we obtain as second bounding functions

$$
\Phi_2(x, x, p, p) = \begin{pmatrix}
\frac{\mu_m(1 - \frac{x_3}{x_{3m}})\sigma_2}{\xi_1 + \xi_2}

\frac{\mu_m(1 - \frac{x_3}{x_{3m}})\sigma_2}{\xi_1 + \xi_2} \Delta_1

\end{pmatrix}
$$

- If $(x_{3m} \in [x_3](t))$ then $1 - \frac{x_3}{x_{3m}}$ has not constant sign on $[x_3](t)$, therefore Rule 1 is not applicable. We must use property P1, i.e. the following decomposition of the specific growth rate $\mu(x_2, x_3) = \mu_1(x_2, k_2, \mu_m) + \mu_2(x_2, x_3, k_3, \mu_m)$, where $\mu_1(x_2, k_2, \mu_m) = \mu_m \frac{x_2}{\xi_1 + \xi_2}$ and $\mu_2(x_2, x_3, k_3, \mu_m) = -\mu_m \frac{x_3 x_2}{x_{3m}(k_2 + x_2)}$, and we obtain the third bounding function

$$
\Phi_3(x, x, p, p) = \begin{pmatrix}
\frac{\mu_m(1 - \frac{x_3}{x_{3m}})\sigma_2}{\xi_1 + \xi_2}

\frac{\mu_m(1 - \frac{x_3}{x_{3m}})\sigma_2}{\xi_1 + \xi_2} \Delta_1

\end{pmatrix}
$$
we obtain the following hybrid interval observer for bioreactor (31)

\[
\begin{bmatrix}
\dot{x} \\
\ddot{x}
\end{bmatrix} = \begin{bmatrix}
-D & k_1 & 0 & 0 & 0 & 0 \\
0 & -D + k_2 & 0 & 0 & 0 & 0 \\
\beta & 0 & -D + k_6 & 0 & k_3 & 0 \\
0 & 0 & 0 & -D & k_1 & 0 \\
0 & 0 & 0 & 0 & -D + k_2 & 0 \\
0 & k_3 & 0 & \beta & 0 & -D + k_6 \\
\end{bmatrix}
\begin{bmatrix}
x \\
\ddot{x}
\end{bmatrix} + \begin{bmatrix}
\Phi_3(x, \dot{x}, \ddot{x}, p, p) \\
\Phi_4(x, \dot{x}, \ddot{x}, p)
\end{bmatrix} + \begin{bmatrix}
y_1(t) + b_1(t) \\
y_2(t) + b_2(t) \\
y_1(t) - b_1(t) \\
y_2(t) - b_2(t)
\end{bmatrix},
\]

where \( q \in \{1, 2, 3\} \) and \( k_2, k_6 \) must be negative. The switching sequence \( \sigma \) is generated according to the sign of \( 1 - \frac{x_3(t)}{x_3} \), when \( x_3(t) \) is taken in \([x_3]\).
We have introduced a new method for designing interval observers for uncertain nonlinear dynamical systems based on an improved version of our hybrid method for computing an over-approximation of reachable sets. We have also proved the positivity of the observation error and its practical stability thanks to a suitable tuning of the observation gain matrix. Another contribution of the work consists in replacing the cooperativity, stability and boundedness assumptions by an unique practical stability condition. In future works, we plan to use this type of interval observer for model-based fault detection.

APPENDIX

II. An adapted version of Müller’s theorem

The theorem below presents an adapted version of Müller’s theorem as given in [32]. It is given here for completeness.

**Theorem 2 (Müller’s theorem [31, 32, 37])** If $\tau_i(t)$ and $\bar{\tau}_i(t)$ satisfy the following inequalities for all $i \in \{1, \ldots, n\}$

- the left Dini derivatives $D^-x_i(t)$ and $D^-\tau_i(t)$ and the right Dini derivatives $D^+x_i(t)$ and $D^+\tau_i(t)$ of $\tau_i(t)$ and $\bar{\tau}_i(t)$ are such that

$$D^+\tau_i(t) \leq \min_{D(t)} f_i(x, p, t)$$

$$D^+\bar{\tau}_i(t) \geq \max_{\bar{D}(t)} f_i(x, p, t)$$

where $D(t)$ is the subset of $\mathbb{D}(t)$ defined by

$$\mathbb{D}_i : \begin{cases} x_i = \tau_i(t) \\
\tau_j(t) \leq x_j \leq \bar{\tau}_j(t), j \neq i \\
p \leq p \leq \bar{p}
\end{cases}$$

and where $\bar{D}(t)$ is the subset of $\bar{D}(t)$ defined by

$$\bar{\mathbb{D}}_i : \begin{cases} x_i = \bar{\tau}_i(t) \\
\tau_j(t) \leq x_j \leq \bar{\tau}_j(t), j \neq i \\
p \leq p \leq \bar{p}
\end{cases}$$

Then for all $x_0 \in [\tau_0, \bar{\tau}_0]$, $p \in [\mathbf{p}, \bar{p}]$, the solution $x(t)$ of system (1) exists and remains bounded by

$$x(t) \leq x(t) \leq \bar{x}(t), \quad \forall t \geq t_0$$

In addition, if for all $p \in [\mathbf{p}_0, \bar{p}_0]$, function $f(x, p, t)$ is Lipschitz with respect to $x$ over $\mathbb{D}$, then this solution is unique for any given $p$.

III. PROOF OF THE MAIN RESULTS

In this Appendix, we prove the results stated in Section 3, namely the positivity and the convergence of the proposed hybrid interval observer.

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III.1. Proof of positivity

Before proving the positivity of the observation error, let us state the following propositions.

**Proposition 3 (P3)** Let $A$ be an arbitrary square real matrix of order $n$ and
- $\mathcal{A}$ is its associated square matrix of order $2n$ obtained according to definition 3,
- $\mathcal{A}$ is the associated square matrix to $\mathcal{A}$ according to definition 2.

Then (49) becomes

$$
\tilde{R} = \begin{bmatrix} \mathcal{A} \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} + \begin{bmatrix} -A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x} \end{bmatrix} = \mathcal{A} \begin{bmatrix} \mathbf{x} - \mathbf{x} \\ \mathbf{x} - \mathbf{x} \end{bmatrix},
$$

(48)

where $\mathcal{A}$ and $\mathcal{A}$ are submatrices of $\mathcal{A}$ with dimension $n \times 2n$ such that $\mathcal{A} = \begin{bmatrix} \mathcal{A} \\ \mathcal{A} \end{bmatrix}$.

**Proof.** For any $i \in \{1, \ldots, n\}$, we have from the left side of (48) the following expression

$$
\mathcal{A}_i \mathbf{x}_i + \sum_{j \in \mathcal{S}} \mathcal{A}_{ij} \mathbf{x}_j + \sum_{j=1}^{n} \mathcal{A}_{i(n+j)} \mathbf{x}_j - A_{ii} \mathbf{x}_i - \sum_{j \in \mathcal{S}} A_{ij} \mathbf{x}_j
$$

(49)

where $\mathcal{S} = \{1, \ldots, i-1, i+1, \ldots, n\}$, $\mathcal{A}_{ii} = A_{ii}$, and $\forall j \in \mathcal{S}$, $\mathcal{A}_{ij} \geq 0$ and $\mathcal{A}_{i(n+j)} \leq 0$. We define by $\mathcal{S}_1$ the subset of $\mathcal{S}$ such that for all $j \in \mathcal{S}_1$

$$
\mathcal{A}_{ij} \neq 0 \text{ and } \mathcal{A}_{ij} = A_{ij} > 0.
$$

In the same way, we define by $\mathcal{R}_1$ the subset of $\{\mathcal{S} - \mathcal{S}_1\}$ such that for all $j \in \mathcal{R}_1$

$$
\mathcal{A}_{i(n+j)} \neq 0 \text{ and } \mathcal{A}_{i(n+j)} = A_{ij} < 0.
$$

Then (49) becomes

$$
\mathcal{A}_i (\mathbf{x}_i - \mathbf{x}_i) + \sum_{j \in \mathcal{S}_1} \mathcal{A}_{ij} (\mathbf{x}_j - \mathbf{x}_j) + \sum_{j \in \mathcal{R}_1} -\mathcal{A}_{i(n+j)} (\mathbf{x}_j - \mathbf{x}_j).
$$

Moreover, since for all $j \in \mathcal{S}$, $\mathcal{A}_{ij} \geq 0$ and $\mathcal{A}_{i(n+j)} \leq 0$, we obtain

$$
\mathcal{A}_i (\mathbf{x}_i - \mathbf{x}_i) + \sum_{j \in \mathcal{S}_1} |\mathcal{A}_{ij}| (\mathbf{x}_j - \mathbf{x}_j) + \sum_{j \in \mathcal{R}_1} |\mathcal{A}_{i(n+j)}| (\mathbf{x}_j - \mathbf{x}_j),
$$

which is exactly the $i^{th}$ line of the right side of (48). Similarity we can prove this equality for any $i \in \{n+1, \ldots, 2n\}$. This completes the proof. \(\square\)

**Definition 5.** For an arbitrary real $n \times m$ matrix $A$, let $B = |A|$ be the $n \times m$ matrix defined by

$$
b_{ij} = |a_{ij}|, \quad \forall i \in \{1, \ldots, n\}, \quad \forall j \in \{1, \ldots, m\}.
$$

(50)

**Proposition 4 (P4)** Let $-K$ be an $n \times m$ arbitrary real matrix, and $\mathcal{X}$ its associated real matrix with dimension $2n \times 2m$ obtained according to definition 4. Then the following equality holds

$$
\begin{bmatrix} \mathcal{X} \\ -\mathcal{X} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & -K \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{y} \end{bmatrix} = |\mathcal{X}| \begin{bmatrix} \mathbf{y} - \mathbf{y} \\ \mathbf{y} - \mathbf{y} \end{bmatrix}
$$

(51)

where $\mathcal{X}$ and $\mathcal{X}$ are submatrices of $\mathcal{X}$ with dimension $n \times 2m$. 

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Proof. We can prove this proposition with the same reasoning as the one in the proof of Proposition P3 except that we set \( S = \{1, \ldots, n\} \).

Now, let us define by \( \mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}(t) \) and \( e(t) = \mathbf{x}(t) - \mathbf{x}(t) \) respectively the upper and the lower observation errors, where \( \mathbf{x}(t) \) and \( \mathbf{\bar{x}}(t) \) are the solutions of the hybrid coupled observer (27) and \( \mathbf{x}(t) \) is the actual solution of (20). Then, we will show that for all \( t \geq t_0 \), both errors are positive for any initial state vector \( \mathbf{x}_0 \in [\mathbf{x}_0, \mathbf{\bar{x}}_0] \), and any vector of parameters \( \mathbf{p} \in [\mathbf{p}] \).

Thanks to Proposition P3 and Proposition P4, and by simple computation we can describe the dynamics of the observation errors by

\[
\begin{bmatrix}
\dot{\mathbf{e}} \\
\mathbf{\bar{e}}
\end{bmatrix} = \hat{\mathbf{e}}(t) \begin{bmatrix}
\mathbf{\bar{e}} \\
\mathbf{e}
\end{bmatrix} + \begin{bmatrix}
\mathbf{\bar{e}}(\mathbf{x}, \mathbf{\bar{x}}, \mathbf{\bar{p}}, \mathbf{\bar{p}}, t) - \Phi_{\mathbf{q}}(\mathbf{x}, \mathbf{p}, t) \\
\Phi_{\mathbf{q}}(\mathbf{x}, \mathbf{p}, t) - \Phi_{\mathbf{q}}(\mathbf{x}, \mathbf{\bar{x}}, \mathbf{\bar{p}}, \mathbf{\bar{p}}, t)
\end{bmatrix} + |\mathbf{\mathcal{X}}_q| \begin{bmatrix}
\mathbf{b}(t) \\
\mathbf{\bar{b}}(t)
\end{bmatrix},
\]

where \( q \in L = \{1, \ldots, l\} \) and

\[
\hat{\mathbf{e}}(t) = \hat{\mathbf{e}}(t) + \mathbf{\mathcal{E}}(t).
\]

Then, the dynamics of the observation errors has the following properties: \( \forall \mathbf{x}_0 \in [\mathbf{x}_0] \), \( \forall \mathbf{p} \in [\mathbf{p}] \), \( \forall q \in L \) and \( \forall t \geq t_0 \)

- the matrices \( \hat{\mathbf{e}}(t) \) are Metzler,
- all the entries of matrices \( |\mathbf{\mathcal{X}}_q| \) are positive,
- in line with Rule 1, Property P1 and Proposition P2, one has

\[
\mathbf{\mathcal{F}}_{\mathbf{q}}(\mathbf{x}, \mathbf{\bar{x}}, \mathbf{\bar{p}}, \mathbf{\bar{p}}, t) - \Phi_{\mathbf{q}}(\mathbf{x}, \mathbf{p}, t) \geq 0 \quad \text{and} \quad \Phi_{\mathbf{q}}(\mathbf{x}, \mathbf{p}, t) - \Phi_{\mathbf{q}}(\mathbf{x}, \mathbf{\bar{x}}, \mathbf{\bar{p}}, \mathbf{\bar{p}}, t) \geq 0.
\]

As a consequence, the theory of monotone cooperative systems [34] allows us to state that:

- if \( \mathbf{e}(t_0) \geq 0 \) and \( e(t_0) \geq 0 \), then \( \forall \mathbf{x}_0 \in [\mathbf{x}_0] \), \( \forall \mathbf{p} \in [\mathbf{p}] \), and \( \forall t \geq t_0 \) the observation errors \( \mathbf{e}(t) \) and \( e(t) \) remain positive.

As a consequence, we can summarize the proof of positivity in the following proposition.

Proposition 5 (P5) The hybrid interval observer (27) satisfies by construction property (29), for any \( K_q \).

III.2. Proof of convergence

The aim of interval observers is to yield an enclosure of the state trajectories with a convergent size, namely property (30). Then, it is of interest to use the notion of practical stability to obtain conditions on the gain matrices \( K_q \) which guarantee this convergence.

III.2.1. Practical stability for hybrid systems. Here, we will use theorems stated in [29] which give sufficient conditions for \( \varepsilon \)-practical stability of hybrid systems, that is conditions which keep state trajectories within given bounds. We recall these theorems and we show how to use them to compute suitable observation gain matrices \( K_q \) which ensure the practical stability of the observation error.

Definition 6 (\( \varepsilon \)-practical stability over \( \tau \)) Assume that a time interval \( \tau \) and switching law \( \gamma \) over \( \tau \) are given. Given an \( \varepsilon > 0 \), the hybrid system \( \mathcal{H} \) is said to be \( \varepsilon \)-practically stable over \( \tau \) under \( \gamma \) if there exists a \( \delta > 0 \) such that \( \|\mathbf{y}(t)\| \leq \varepsilon \), \( \forall t \in \tau \) whenever \( \|\mathbf{y}(t_0)\| \leq \delta \). Here \( \mathbf{y}(t) \) denotes the continuous state trajectory of \( \mathcal{H} \).
Definition 7 ($\varepsilon$-practical Lyapunov-like function) Given a time interval $\tau$ and switching law $\gamma$ over $\tau$, a continuously differentiable real-valued function $V(y(t), t)$ satisfying $V(0, t) = 0$, $\forall t \in \tau$ is an $\varepsilon$-practical Lyapunov-like function over $\tau$ under $\gamma$ if there exists a Lebesgue integrable function $\varphi(y(t), t)$, positive constants $\mu_1$ and $\mu_2$, such that for any trajectory $y(t)$ generated by $\gamma$ that starts from $y(t_0)$ and its corresponding switching sequence $\sigma = ((t_0, q_0), (t_1, q_1), \ldots, (t_s, q_s), \ldots) \in \Sigma$, $q_s \in L$, the following holds:

1. $V(y(t), t) \leq \varphi(y(t), t)$, a.e. $t \in \tau$;
2. $V(J_{q_0,q_{i+1}}(y(t_0^-)), t_s^-) \leq \mu_1 V(y(t_s^-), t_s^+)$ at any switching instant $t_s$;
3. $\int_{t_0}^{t} \mu_1^N(z) \varphi(y(z), z)dz < \inf_{|x| > \varepsilon} V(y(t), t) - \mu_2 \mu_1^N(h_0(t))$, $\forall t \in \tau$.

$N(t_a, t_b)$ denotes the number of switching during the time interval $\tau$, and reset functions $J_{q_i,q_{i+1}}$ are defined by (9).

Note that $V(\ldots)$ is not a Lyapunov function in the usual sense, since its definiteness condition is not imposed on it or its derivatives.

Theorem 3 ([29]) Given a time interval $\tau$ and switching law $\gamma$ over $\tau$, an hybrid system $\mathcal{H}$ is $\varepsilon$-practically stable over $\tau$ under $\gamma$, if there exists an $\varepsilon$-practical Lyapunov-like function $V(y(t), t)$ over $\tau$ under $\gamma$.

In the next sections, we show how to apply the above results to the dynamics the observation error.

III.2.2. Practical stability of the observation error. First we can easily prove that the dynamics of the total observation error $e(t) = \tilde{e}(t) + e(t) = \tilde{x}(t) - x(t)$ is given by the hybrid system below

$$e = \tilde{E}_q(t)e + \left(\tilde{E}_q(x, \tilde{x}, p, \tilde{p}, t) - \Phi_q(x, \tilde{x}, p, \tilde{p}, t)\right) + |K_q|b(t)$$

where $q \in L = \{1, \ldots, l\}$ and $E_q(t) = A(t) + K_q C(t)$. We can rewrite this hybrid system as follows

$$e = \tilde{E}_q(t)e + h_q(t), \quad q \in L = \{1, \ldots, l\}.$$  

(55)

where $h_q(t) = \Phi_q(z_1, r_1, t) - \Phi_q(z_2, r_2, t) + \|K_q\|b(t)$, and $z_1^T = (x, \tilde{x})$, $z_2^T = (\tilde{x}, x)$, $r_1^T = (p, \tilde{p})$, $r_2^T = (\tilde{p}, p)$.

Since $\Phi(\ldots)$ is Lipschitz continuous w.r.t state and parameter vectors, we denote by $L_\varepsilon(t)$ the Lipschitz constant of $\Phi$ w.r.t $x(t)$ and by $L_p(t)$ its Lipschitz constant w.r.t $p$, and we set $h_q(t) = L_\varepsilon(t)\|z_1 - z_2\| + \|K_q\|\|b(t)\|$. Let $\lambda_0(t) = \max\{\lambda_\mu(t_i, 0)\}$ where $\lambda_\mu(t_i)$ are the eigenvalues of $\tilde{E}_q(t)$.

We can now address the proof of convergence.

Proposition 6 (P6) The hybrid system (55) is $\varepsilon$-practically stable over $\tau$ under $\gamma$ if

$$\forall t \in \tau, \quad \int_{t_0}^{t} \lambda_0(z)dz < \frac{1}{2}$$

(56)

which can be satisfied if there exists some gain matrices $K_q$ such that the matrices $\tilde{E}_q(t)$ are stable. In addition, the lower bound on $\varepsilon$ such that system (55) is $\varepsilon$-practically stable over $\tau$ under $\gamma$ is given by

$$\forall t \in \tau, \quad \varepsilon = w(\|p - \tilde{p}\|, \|b(t)\|) > \frac{(\int_{t_0}^{t} \tilde{h}_q(z)dz + \|p - \tilde{p}\| \int_{t_0}^{t} L_p(z)dz)}{(\frac{1}{2} - \int_{t_0}^{t} \lambda_0(z)dz)}$$

(57)
Proof. Let us consider $V(e,t) = \frac{1}{2}e^T e$. Form (55) we have

$$
\dot{V}(e,t) = e^T \dot{\tilde{E}}_q(t)e + e^T h_q(t)
\leq \lambda_0(t) \Vert e \Vert^2 + \Vert L_s(t) \Vert \Vert z_1 - z_2 \Vert + \Vert K_q\Vert \Vert b(t) + L_p(t) \Vert \Vert p - \tilde{p} \Vert
\leq \lambda_0(t) \Vert e \Vert^2 + \Vert \bar{E}_s(t) + L_p(t) \Vert \Vert p - \tilde{p} \Vert
$$

(58)

Now, we can define piecewise continuous functions $\alpha(t) \equiv \lambda_0(t)$ and $\beta(t) \equiv \bar{E}_s(t) + L_p(t) \Vert p - \tilde{p} \Vert$. For any $\varepsilon(t)$ satisfying $\Vert e(t) \Vert \leq \varepsilon$, (58) leads to $V(e,t) \leq \varepsilon^2 \alpha(t) + \varepsilon \beta(t)$. Let us choose $\varphi(e,t) = \varepsilon^2 \alpha(t) + \varepsilon \beta(t)$ and let $\mu_1 = 1$ because $V(\cdot, \cdot)$ has the same value after and prior to the switching between two bounding subsystems. According to Theorem 3 and definition 7, system (55) is $\varepsilon$-practically stable over $\tau$ under $\gamma$ if the following inequality holds

$$
\exists \mu_2 > 0, \forall t \in \tau, \int_0^t (\varepsilon^2 \alpha(z) + \varepsilon \beta(z))dz < \frac{1}{2} \varepsilon^2 - \mu_2
$$

(59)

which, in accordance with the definitions of $\alpha(t)$ and $\beta(t)$, is equivalent to $\exists \mu_2 > 0, \forall t \in \tau,$

$$
\left( - \frac{1}{2} + \int_0^t \lambda_0(z)dz \right) \varepsilon^2 + (\int_0^t \bar{E}_s(z) + \Vert \tilde{p} - p \Vert \int_0^t L_p(z)dz) \varepsilon < -\mu_2.
$$

(60)

Let us choose $\mu_2 \ll 1$, then we only need to have

$$
\forall t \in \tau, \left( - \frac{1}{2} + \int_0^t \lambda_0(z)dz \right) + (\int_0^t \bar{E}_s(z) + \Vert \tilde{p} - p \Vert \int_0^t L_p(z)dz) < 0.
$$

(61)

Then, since $\varepsilon$ must be positive, the last inequality (61) is equivalent to the requirement (57) which is true if and only if (56) is satisfied. This completes the proof.

Finally, Propositions P5 and P6 prove the main result, i.e. Theorem 1.

REFERENCES


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Figure 1. The automaton which governs the collection of bounding systems $\mathcal{H}$

Figure 2. The state space partition with respect to the guard condition $x_1 = 0$. The blue and green state boxes represent an over-approximation of the reachable set of (16) with $[15, 20] \times [2, 4]$ as initial condition and $T = [0, 20]$ as simulation period. For all state boxes $[x](t)$ where $x_1(t) < 0$, the activated bounding system is $M_1$, and for all state boxes $[x](t)$ where $x_1(t) > 0$, the activated bounding system is $M_2$. Finally for the state boxes $[x](t)$ which contain 0, the activated bounding system is $M_3$.

Figure 3. A zoom around the state region where the reached set of (16) crosses the guard condition. The blue discontinuous rectangular represents the state box when the reached set computed by the bounding system $M_2$ intersects for the first time the guard condition, this event will activate the bounding system $M_3$. On the other hand, the black continuous rectangular represents the state box when the reached set computed by the bounding system $M_3$ leaves completely the guard condition. This event deactivates the use of $M_3$ and simultaneously activates the bounding system $M_1$. Consequently, the green boxes show the sliding of all the state vectors before and after the intersection with the set defined by the guard condition.
Figure 4. Lower and upper estimates of the biomass state variable.
Figure 5. Lower and upper estimates of the substrate state variable.
Figure 6. Lower and upper estimates of the product state variable.
Figure 7. Switching sequence.